



A New DDH Secure Group for Applications of ElGamal in Cryptographic Voting Protocols

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DDH Secure Groups for ElGamal

- ▶ Prime-order subgroup $\mathbb{G}_q \subset \mathbb{Z}_p^*$ of integers modulo p
 - ▶ General case: $p = kq + 1$, for co-factor $k \geq 2$
 - ▶ Safe prime: $p = 2q + 1$
- ▶ Prime-order subgroup $E_q \subseteq E[\mathbb{F}_p]$ of elliptic curve over \mathbb{F}_p
- ▶ Prime-order subgroup $F_q \subset \mathbb{F}_{p^k}^*$ of polynomials of degree k over \mathbb{F}_p

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Subgroup of Integers Modulo Safe Prime

- ▶ The elements of $\mathbb{G}_q \subset \mathbb{Z}_p^*$ modulo $p = 2q + 1$ are *quadratic residues*:

$$\mathbb{G}_q = \{x^2 \bmod p : 1 \leq x < p\}$$

- ▶ Example: $p = 23$, $q = 11$

\mathbb{Z}_{23}^*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
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Properties of \mathbb{G}_q

- ▶ All elements of $\mathbb{G}_q \setminus \{1\}$ are generators
 - ▶ 3 is always a generator
 - ▶ 4, 9, 16, 25, ... are always generators
- ▶ Subgroup membership
 - ▶ Method 1: $x \in \mathbb{G}_q$, iff $x^q \bmod p = 1$
 - ▶ Method 2: $x \in \mathbb{G}_q$, iff $\left(\frac{x}{p}\right) = 1$
- ▶ Quadratic residues vs. quadratic non-residues
 - ▶ $x \in \mathbb{G}_q$ implies $p - x \notin \mathbb{G}_q$
 - ▶ $x \notin \mathbb{G}_q$ implies $p - x \in \mathbb{G}_q$

Practical Disadvantages of \mathbb{G}_q

- ▶ Checking group membership is expensive ($1 \times \text{modexp}$ xor $1 \times \text{Jacobi symbol}$)
- ▶ Group membership depends on p
 - ▶ Selecting generators
 - ▶ Generating random group elements
- ▶ ElGamal with message space \mathbb{G}_q
 - ▶ Mapping $\Gamma : \{0, 1\}^n \rightarrow \mathbb{G}_q$ for general-purpose messages $M \in \{0, 1\}^n$ depends on p
 - ▶ Mapping $\Gamma : \mathcal{M} \rightarrow \mathbb{G}_q$ for specific messages $M \in \mathcal{M}$ depends on p
 - ▶ Example: prime number encoding of votes

Proposal For a New Group

- ▶ Computing absolute values in \mathbb{Z}_p^*

$$\text{abs}(x) \stackrel{\text{def.}}{=} \begin{cases} x, & \text{if } 1 \leq x \leq q \\ p - x, & \text{if } q < x < p \end{cases}$$

- ▶ Let $\mathbb{Z}_p^+ = \{\text{abs}(x) : x \in \mathbb{Z}_p^*\} = \{1, \dots, q\}$



- ▶ Let $x \circ y = \text{abs}(xy \bmod p)$ and $\text{inv}(x) = \text{abs}(x^{-1} \bmod p)$
- ▶ $(\mathbb{Z}_p^+, \circ, \text{inv}, 1)$ forms a group, the group of absolute values modulo $p = 2q + 1$

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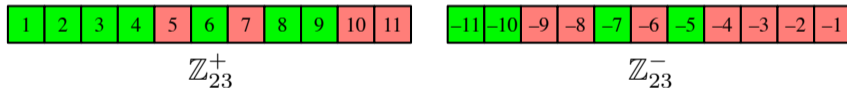
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Properties of \mathbb{Z}_p^+

- ▶ Exponentiations in \mathbb{Z}_p^+ can be computed efficiently as $\text{abs}(x^y \bmod p)$
- ▶ From $|\mathbb{Z}_p^+| = |\mathbb{G}_q| = q$, it follows that \mathbb{Z}_p^+ is isomorphic to \mathbb{G}_q
- ▶ The isomorphism can be computed efficiently in both directions
 - ▶ $\phi(x) = x^2 \bmod p$, for $x \in \mathbb{Z}_p^+$
 - ▶ $\phi^{-1}(y) = \text{abs}(\sqrt{y} \bmod p) = \text{abs}(y^{\frac{q+1}{2}})$, for $y \in \mathbb{G}_q$
- ▶ Therefore, if DDH is hard in \mathbb{G}_q , it is equally hard in \mathbb{Z}_p^+

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Practical Advantages of \mathbb{Z}_p^+ Over \mathbb{G}_q

- ▶ Group membership $x \in \mathbb{Z}_p^+$ can be tested efficiently as $1 \leq x \leq q$
- ▶ Since $p < p'$ implies $\mathbb{Z}_p^+ \subset \mathbb{Z}_{p'}^+$, it follows that:
 - ▶ $1, 2, 3, 4, 5, \dots$ are always group elements,
 - ▶ $1, 2, 3, 4, 5, \dots$ are possible random group elements,
 - ▶ $2, 3, 4, 5, 6, \dots$ are always generators,independently of p
- ▶ General-purpose messages $M \in \{0, 1\}^n$ can be mapped into \mathbb{Z}_p^+ by interpreting them as binary numbers (except for 0)
- ▶ For specific messages, $\Gamma : \mathcal{M} \rightarrow \mathbb{Z}_p^+$ can be defined independently of p

Conclusion

- ▶ From a security perspective, \mathbb{Z}_p^+ and \mathbb{G}_q are equivalent (DDH holds)
- ▶ Group operation in \mathbb{Z}_p^+ is slightly less efficient (but the cost is negligible)
- ▶ Membership tests in \mathbb{Z}_p^+ are much more efficient
- ▶ Plus some other practical advantages
- ▶ General recommendation:

Use \mathbb{Z}_p^+ instead of \mathbb{G}_q in applications and implementations of ElGamal

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